Compliant Wall Analysis of an Electrically Conducting Jeffrey Fluid with Peristalsis

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This investigation looks at the peristaltic flow of a magnetohydrodynamic (MHD) Jeffrey fluid in a channel with compliant walls. The flow induced is due to sinusoidal waves on the channel walls. A series solution of the resulting boundary value problem is derived when the wave amplitude is small. Effects of various interesting flow parameters are discussed with the help of graphs.

Key words: Peristaltic Flow; Jeffrey Fluid; MHD; Compliant Walls.

1. Introduction

The motion of a fluid in a channel or pipe due to a sequence of waves which introduces clasping and compressing in the boundary walls is known as peristaltic motion. The study of peristaltic flow of non-Newtonian fluids [1 – 5] has attracted the attention of many researchers in the recent years. There are various applications in physiology where this motion is used by the body to propel or mix the contents of a tube. It appears to be the major mechanism for urine transport in ureter, food mixing and chyme movement in the gastrointestinal tract, transport of spermatozoa in cervical canal, transport of bile in bile ducts and so on. Technical roller and finger pumps also operate according to this rule. The literature on peristaltic motion in a viscous fluid is abundant and its complete survey in the domain of biomechanics has been given by Jaffrin and Shapiro [6]. However, it is an established fact now that many of the physiological fluids are non-Newtonian.

The study of biomagnetic fluids has attracted the attention of recent researchers. In fact, such attraction is due to the extensive applications of biomagnetic fluids in bioengineering and medical sciences. Special relevance of such fluids appears in the development of magnetic devices for cell separation, cancer tumor treatment causing magnetic hyperthermia, reduction of bleeding during surgeries and many others. The most characteristic biomagnetic fluid is blood. Blood is a biomagnetic fluid due to the complex interaction of intercellular protein, cell membrane, and hemoglobin. On the other hand the motion of a magnetohydrodynamic (MHD) fluid across the magnetic field induces electric currents which change the magnetic field and the flow of the fluid [7]. The MHD flow in a channel having an elastic, rhythmically contracting wall (peristaltic flow) is of great interest in connection with certain problems of the movement of conductive physiological fluids (e.g. the blood and blood pump machines). Sud et al. [8] studied the influence of a moving magnetic field on blood flow. The blood flow in the context of magnetohydrodynamics is also investigated by Srivastava and Agrawal [9]. The MHD blood flow through an equally branched channel with flexible boundaries inducing peristaltic waves has been examined by Agrawal and Anwaruddin [10]. Now the MHD peristaltic flows of viscous and non-Newtonian fluids under different aspects have been analyzed in details (the interested readers may see the studies [11] and several references there in). Some recent analytic and numerical solutions for the peristaltic flows can be seen in the investigations [12 – 27].

Literature survey witnesses that very little attention has been given to the peristaltic motion with compliant walls. Such consideration explains how a progressive wave may be imparted to the walls. The compliant wall is excited by the muscles whose tension controls its deformation. Such action of muscles can be described by equations in terms of compliant wall displacement [28]. Abd Elnaby and Haroun [29] discussed the peristaltic flow of a viscous fluid in a two-dimensional channel with compliant walls. They presented the flow analysis under the assumption of small amplitude ratio. Later Ali et al. [30] analyzed the peristaltic flow of a Maxwell fluid in a channel with compliant walls. The objective of this study is to analyze the peristaltic flow of non-Newtonian fluids in a channel with compliant walls. For that the constitutive equations of a linear Jeffrey fluid have been used. The considered fluid model is simplest and can describe the rheological characteristics in terms of relaxation and retardation time parameters. The paper is organized as follows. In Section 2, we provide the mathematical modelling and problem statement. The nonlinear differential equations subject to appropriate boundary conditions are solved analytically in Section 3. Section 4 describes the graphical results. Closing remarks are presented in Section 5.

2. Problem Definition

We consider a two-dimensional channel of uniform width $2h$ filled with a homogenous Jeffrey fluid. The fluid motion in the channel is induced by imposing small amplitude sinusoidal waves on the compliant walls of the channel. The physical model of the problem is given as follows:

The geometry of the walls are given by

$$y = \pm h \pm \eta \tag{1}$$

in which the vertical displacement η of the upper wall from its normal position is defined as

$$\eta = a\cos\frac{2\pi}{\lambda}(x-ct), \tag{2}$$

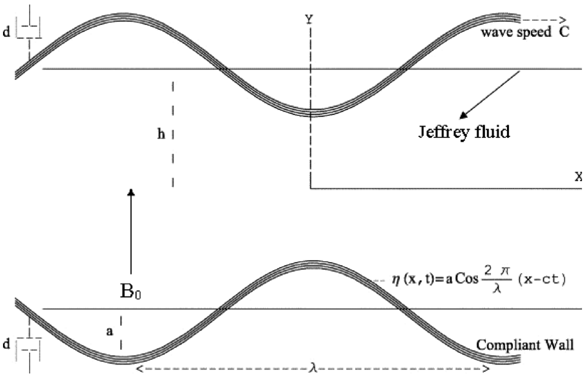

Fig. 1. Scheme of the problem.

where a is the wave amplitude, c is the constant wave speed, λ is the wavelength, for lower wall η is replaced by $-\eta$, and in the Cartesian coordinate system x is measured in the direction of wave propagation and y is measured in the direction normal to the x-axis. The relation describing the compliant nature of the wall is given by

$$\left[m\frac{\partial^2}{\partial t^2} + d\frac{\partial}{\partial t} + B\frac{\partial^4}{\partial x^4} - T\frac{\partial^2}{\partial x^2} + K\right]\eta = p - p_0. \tag{3}$$

In above equation m denotes the plate mass per unit area, d is the wall damping coefficient, B is the flexural rigidity of the plate, T is the longitudinal tension per unit width, K is the spring stiffness and p_0 is the pressure on the outside surface of the wall. We assume $p_0 = 0$ and the channel walls are inextensible so that only their lateral motions normal to the undeformed positions occur. The horizontal displacement of the walls is taken zero and hence the boundary conditions are

$$\Psi_y = 0 \text{ and } \Psi_x = \mp\frac{\partial\eta}{\partial t} \text{ at } y = \pm h \pm \eta, \tag{4}$$

where the stream function $\Psi(x,y,t)$ is defined as

$$u = \frac{\partial\Psi}{\partial y}, \quad v = -\frac{\partial\Psi}{\partial x}. \tag{5}$$

Here u and v are the x- and y-components of the velocity $\mathbf{V}$, respectively.

The fundamental equations describing the MHD flow of an incompressible fluid are

$$\operatorname{div}\mathbf{V} = 0, \tag{6}$$

$$\rho\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = -\nabla p + \operatorname{div}\mathbf{S} - \sigma B_0^2\mathbf{V}, \tag{7}$$

with p as the pressure, $\mathrm{d}/\mathrm{d}t$ the material time derivative, $\mathbf{S}$ an extra stress tensor which for a Jeffrey fluid is

$$(1+\lambda_1)\mathbf{S} = 2\mu\left(1+\lambda_2\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{A}_1. \tag{8}$$

It should be pointed out that (7) holds when an induced magnetic field is neglected and B_0 is an applied magnetic field. In above equation μ is the dynamic viscosity, λ_1 is the ratio of relaxation to retardation times, λ_2 is the relaxation time, and $\mathbf{A}_1$ is the first Rivlin-Erickson tensor.

Equations (7) and (8) in scalar form can be written as

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)= -\frac{\partial p}{\partial x}+\frac{\partial S_{xx}}{\partial x}+\frac{\partial S_{xy}}{\partial y}-\sigma B_0^2 u, \tag{9}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\frac{\partial S_{xy}}{\partial x}+\frac{\partial S_{yy}}{\partial y}, \tag{10}$$

$$S_{xx}=\frac{2\mu}{1+\lambda_1}\left[1+\lambda_2\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\right]\frac{\partial u}{\partial x}, \tag{11}$$

$$S_{xy}=\frac{\mu}{1+\lambda_1}\left[1+\lambda_2\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\right] \cdot\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), \tag{12}$$

$$S_{yy}=\frac{2\mu}{1+\lambda_1}\left[1+\lambda_2\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\right]\frac{\partial v}{\partial y}. \tag{13}$$

The complete formulation of the problem allows the continuity of stresses which requires that at the interfaces of the walls and the fluid p must be the same as that which acts on the fluid at $y=\pm h\pm\eta$. Employing (9), we have

$$\frac{\partial}{\partial x}\left[m\frac{\partial^2\eta}{\partial t^2}+d\frac{\partial\eta}{\partial t}+B\frac{\partial^4\eta}{\partial x^4}-T\frac{\partial^2\eta}{\partial x^2}+K\eta\right]= -\rho\left(\Psi_{yt}+\Psi_y\Psi_{yx}-\Psi_x\Psi_{yy}\right)+S_{xx,x}+S_{xy,y}-\sigma B_0^2\Psi_y, \tag{14}$$

in which subscripts indicate partial differentiation and the incompressibility condition (6) is identically satisfied through (5).

Introducing the following dimensionless quantities

$$\hat{x}=\frac{x}{h},\ \hat{y}=\frac{y}{h},\ \hat{u}=\frac{u}{c},\ \hat{v}=\frac{v}{c},\ \hat{t}=\frac{ct}{h}, \quad \hat{p}=\frac{p}{\rho c^2},\ \hat{\eta}=\frac{\eta}{h},\ \hat{\Psi}=\frac{\Psi}{ch},\ \hat{m}=\frac{m}{\rho h}, \quad \hat{d}=\frac{dh}{\rho\nu},\ \hat{B}=\frac{B}{\rho h\nu^2},\ \hat{T}=\frac{Th}{\rho\nu^2},\ \hat{K}=\frac{Kh^3}{\rho\nu^2}, \quad \hat{S}=\frac{hS}{\mu c},\ \hat{\lambda}_2=\frac{c}{h}\lambda_2, \tag{15}$$

the resulting problem statement after suppressing hats take the form as

$$\frac{\partial}{\partial t}\nabla^2\Psi+\Psi_y\nabla^2\Psi_x-\Psi_x\nabla^2\Psi_y= \frac{1}{R}\left(S_{xx,xy}+S_{xy,yy}-S_{xy,xx}-S_{yy,xy}\right)-M^2\Psi_{yy}, \tag{16}$$

$$S_{xx}=\frac{2}{1+\lambda_1}\left(\Psi_{xy}+\lambda_2(\Psi_{xyt}+\Psi_y\Psi_{xxy}-\Psi_x\Psi_{xyy})\right), \tag{17}$$

$$S_{xy}=\frac{1}{1+\lambda_1}\left((\Psi_{yy}-\Psi_{xx})+\lambda_2(\Psi_{yyt}+\Psi_y\Psi_{xyy}-\Psi_x\Psi_{yyy} -\Psi_{xxt}-\Psi_y\Psi_{xxx}+\Psi_x\Psi_{xxy})\right), \tag{18}$$

$$S_{yy}=\frac{2}{1+\lambda_1}\left(-\Psi_{xy}+\lambda_2\left(-\Psi_{xyt}-\Psi_y\Psi_{xxy}+\Psi_x\Psi_{xyy}\right)\right), \tag{19}$$

$$\eta=\varepsilon\cos\alpha\,(x-t), \tag{20}$$

$$\Psi_y=0 \quad \text{at } y=\pm1\pm\eta, \tag{21}$$

$$\frac{\partial}{\partial x}\left(m\frac{\partial^2\eta}{\partial t^2}+\frac{d}{R}\frac{\partial\eta}{\partial t}+\frac{B}{R^2}\frac{\partial^4\eta}{\partial x^4}-\frac{T}{R^2}\frac{\partial^2\eta}{\partial x^2}+\frac{K}{R^2}\eta\right)= \frac{1}{R}\left(S_{xx,x}+S_{xy,y}\right)-\left(\Psi_{yt}+\Psi_y\Psi_{yx}-\Psi_x\Psi_{yy}\right)-M^2\Psi_y \quad \text{at } y=\pm1\pm\eta, \tag{22}$$

where $\varepsilon=a/h$ is the amplitude ratio, $\alpha=2\pi h/\lambda$ is the wave number, $R=ch/\nu$ is the Reynolds number, and $\nu=(\mu/\rho)$ is the kinematic viscosity.

3. Method of Solution

Adopting a similar approach as in [31], the flow quantities in powers of ε can be expanded as

$$\Psi=\Psi_0+\varepsilon\Psi_1+\varepsilon^2\Psi_2+\dots, \tag{23}$$

$$\frac{\partial p}{\partial x}=\left(\frac{\partial p}{\partial x}\right)_0+\varepsilon\left(\frac{\partial p}{\partial x}\right)_1+\varepsilon^2\left(\frac{\partial p}{\partial x}\right)_2+\dots, \tag{24}$$

$$S_{xx}=S_{xx0}+\varepsilon S_{xx1}+\varepsilon^2 S_{xx2}+\dots, \tag{25}$$

$$S_{xy}=S_{xy0}+\varepsilon S_{xy1}+\varepsilon^2 S_{xy2}+\dots, \tag{26}$$

$$S_{yy}=S_{yy0}+\varepsilon S_{yy1}+\varepsilon^2 S_{yy2}+\dots. \tag{27}$$

Note that the first term on the right-hand side in (24) corresponds to the imposed pressure gradient and the other terms correspond to the peristaltic motion. Using the above equations into (16)–(22) and then collecting terms of like powers of ε, we obtain three sets of coupled differential equations with the corresponding boundary conditions in ε_0, ε_1, and ε_2.

In actual practice the ε_0 set of differential equations subject to a steady parallel flow and transverse symmetry assumption for a constant pressure gradient in the x-direction corresponds to the classical Poiseuille flow, i.e.

$$\Psi_0(y) = \frac{2K_0}{RM^2}\left[y - \frac{\sinh \Gamma y}{\Gamma \cosh \Gamma}\right] + c_1, \tag{28}$$
$$K_0 = -\frac{R}{2}\left(\frac{dp}{dx}\right)_0,$$

in which $\Gamma = M\sqrt{R(1+\lambda_1)}$ and c_1 is an arbitrary constant.

The second and third sets of differential equations in Ψ_1 and Ψ_2 can be written in the forms:

$$\Psi_1(x,y,t) = \frac{1}{2}\left[\Phi_1(y)\,\mathrm{e}^{\mathrm{i}\alpha(x-t)} + \Phi_1^*(y)\,\mathrm{e}^{-\mathrm{i}\alpha(x-t)}\right], \tag{29}$$

$$S_{xx1} = \frac{1}{2}\left[\Phi_2(y)\,\mathrm{e}^{\mathrm{i}\alpha(x-t)} + \Phi_2^*(y)\,\mathrm{e}^{-\mathrm{i}\alpha(x-t)}\right], \tag{30}$$

$$S_{xy1} = \frac{1}{2}\left[\Phi_3(y)\,\mathrm{e}^{\mathrm{i}\alpha(x-t)} + \Phi_3^*(y)\,\mathrm{e}^{-\mathrm{i}\alpha(x-t)}\right], \tag{31}$$

$$S_{yy1} = \frac{1}{2}\left[\Phi_4(y)\,\mathrm{e}^{\mathrm{i}\alpha(x-t)} + \Phi_4^*(y)\,\mathrm{e}^{-\mathrm{i}\alpha(x-t)}\right], \tag{32}$$

$$\Psi_2(x,y,t) = \frac{1}{2}\Big[\Phi_{20}(y) + \Phi_{22}(y)\,\mathrm{e}^{2\mathrm{i}\alpha(x-t)} + \Phi_{22}^*(y)\,\mathrm{e}^{-2\mathrm{i}\alpha(x-t)}\Big], \tag{33}$$

$$S_{xx2} = \frac{1}{2}\Big[\Phi_{30}(y) + \Phi_{33}(y)\,\mathrm{e}^{2\mathrm{i}\alpha(x-t)} + \Phi_{33}^*(y)\,\mathrm{e}^{-2\mathrm{i}\alpha(x-t)}\Big], \tag{34}$$

$$S_{xy2} = \frac{1}{2}\Big[\Phi_{40}(y) + \Phi_{44}(y)\,\mathrm{e}^{2\mathrm{i}\alpha(x-t)} + \Phi_{44}^*(y)\,\mathrm{e}^{-2\mathrm{i}\alpha(x-t)}\Big], \tag{35}$$

$$S_{yy2} = \frac{1}{2}\Big[\Phi_{50}(y) + \Phi_{55}(y)\,\mathrm{e}^{2\mathrm{i}\alpha(x-t)} + \Phi_{55}^*(y)\,\mathrm{e}^{-2\mathrm{i}\alpha(x-t)}\Big], \tag{36}$$

where the asterisk shows the complex conjugate. Invoking above equations into the differential equations and their corresponding boundary conditions in Ψ_1 and Ψ_2, we have three sets of coupled linear differential equations with their corresponding boundary conditions which are fourth-order ordinary differential equations with variable coefficients, and the boundary conditions are not all homogeneous and the problem is not an eigenvalue problem. For the free pumping case $(\partial p/\partial x)_0 = 0$ which means $K_0 = 0$, we have

$$\mathrm{i}\alpha R\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2\right]\Phi_1(y) = RM^2\Phi_1''(y) + \mathrm{i}\alpha\Phi_4'(y) - \Phi_3''(y) - \alpha^2\Phi_3(y) - \mathrm{i}\alpha\Phi_2'(y), \tag{37}$$

$$(1+\lambda_1)\,\Phi_2(y) = 2\mathrm{i}\alpha(1-\mathrm{i}\alpha\lambda_2)\,\Phi_1'(y), \tag{38}$$

$$(1+\lambda_1)\,\Phi_3(y) = (1-\mathrm{i}\alpha\lambda_2)\left(\Phi_1''(y) + \alpha^2\Phi_1(y)\right), \tag{39}$$

$$(1+\lambda_1)\,\Phi_4(y) = -2\mathrm{i}\alpha(1-\mathrm{i}\alpha\lambda_2)\,\Phi_1'(y), \tag{40}$$

$$\Phi_1'(\pm 1) = 0, \tag{41}$$

$$\mathrm{i}\alpha R\Phi_1'(\pm 1) + \mathrm{i}\alpha\Phi_2(\pm 1) + \Phi_3'(\pm 1) - RM^2\Phi_1'(\pm 1) = R\delta, \tag{42}$$
$$\delta = -\frac{\mathrm{i}\alpha}{R^2}\left(\alpha^2R^2m + \mathrm{i}\alpha Rd - \alpha^4B - \alpha^2T - K\right),$$

$$\Phi_{40}''(y) = \frac{\mathrm{i}\alpha R}{2}\left[\Phi_1^*(y)\,\Phi_1''(y) - \Phi_1(y)\,\Phi_1^{*\prime\prime}(y)\right]' + RM^2\Phi_{20}''(y), \tag{43}$$

$$\Phi_{30}(y) = -\frac{\alpha^2\lambda_2}{1+\lambda_1}\left[\Phi_1(y)\,\Phi_1^{*\prime\prime}(y) + \Phi_1^*(y)\,\Phi_1''(y) + 2\Phi_1'(y)\,\Phi_1^{*\prime}(y)\right], \tag{44}$$

$$\Phi_{40}(y) = \frac{1}{1+\lambda_1}\Phi_{20}''(y) - \frac{\mathrm{i}\alpha\lambda_2}{2(1+\lambda_1)}\left[\Phi_1(y)\,\Phi_1^{*\prime\prime}(y) - \Phi_1^*(y)\,\Phi_1''(y)\right]', \tag{45}$$

$$\Phi_{50}(y) = \frac{\alpha^2\lambda_2}{1+\lambda_1}\left[\Phi_1(y)\,\Phi_1^{*\prime\prime}(y) + \Phi_1^*(y)\,\Phi_1''(y) + 2\Phi_1'(y)\,\Phi_1^{*\prime}(y)\right], \tag{46}$$

$$\Phi_{20}'(\pm 1) = \mp\frac{1}{2}\left[\Phi_1''(\pm 1) + \Phi_1^{*\prime\prime}(\pm 1)\right], \tag{47}$$

$$\begin{aligned}\Phi_{40}'(\pm 1) = &-\frac{\mathrm{i}\alpha R}{2}\left[\Phi_1(\pm 1)\,\Phi_1^{*\prime\prime}(\pm 1) - \Phi_1^*(\pm 1)\,\Phi_1''(\pm 1)\right] + RM^2\Phi_{20}'(\pm 1)\\ &\mp\frac{\mathrm{i}\alpha R}{2}\left[\Phi_1''(\pm 1) - \Phi_1^{*\prime\prime}(\pm 1)\right]\\ &\pm\frac{RM^2}{2}\left[\Phi_1''(\pm 1) + \Phi_1^{*\prime\prime}(\pm 1)\right]\\ &\mp\frac{\mathrm{i}\alpha}{2}\left[\Phi_2'(\pm 1) - \Phi_2^{*\prime}(\pm 1)\right]\\ &\mp\frac{1}{2}\left[\Phi_3''(\pm 1) + \Phi_3^{*\prime\prime}(\pm 1)\right],\end{aligned} \tag{48}$$

$$2\mathrm{i}\alpha R\left[\frac{d^2}{dy^2}-4\alpha^2\right]\Phi_{22}(y)=RM^2\Phi_{22}''(y)+\frac{\mathrm{i}\alpha R}{2}\left[\Phi_1'(y)\,\Phi_1''(y)-\Phi_1(y)\,\Phi_1'''(y)\right]-\Phi_{44}''(y)-4\alpha^2\Phi_{44}(y)-2\mathrm{i}\alpha\Phi_{33}'(y)+2\mathrm{i}\alpha\Phi_{55}'(y),\quad(49)$$

$$(1+\lambda_1)\,\Phi_{33}(y)=\frac{\alpha^2\lambda_2}{(1+\lambda_1)}\left[\Phi_1(y)\,\Phi_1''(y)-\Phi_1'^2(y)\right]+\frac{4\mathrm{i}\alpha(1-2\mathrm{i}\alpha\lambda_2)}{(1+\lambda_1)}\Phi_{22}'(y),\quad(50)$$

$$(1+\lambda_1)\,\Phi_{44}(y)=\frac{\mathrm{i}\alpha\lambda_2}{2}\left[\Phi_1'(y)\,\Phi_1''(y)-\Phi_1(y)\,\Phi_1'''(y)\right]+\frac{(1-2\mathrm{i}\alpha\lambda_2)}{(1+\lambda_1)}\Phi_{22}''(y)\cdot\frac{4\alpha^2(1-2\mathrm{i}\alpha\lambda_2)}{(1+\lambda_1)}\Phi_{22}(y),\quad(51)$$

$$(1+\lambda_1)\,\Phi_{55}(y)=-\frac{\alpha^2\lambda_2}{(1+\lambda_1)}\left[\Phi_1(y)\,\Phi_1''(y)-\Phi_1'^2(y)\right]-\frac{4\mathrm{i}\alpha(1-2\mathrm{i}\alpha\lambda_2)}{(1+\lambda_1)}\Phi_{22}'(y),\quad(52)$$

$$\Phi_{22}'(\pm1)=\mp\frac{1}{2}\Phi_1''(\pm1),\quad(53)$$

$$2\Phi_{44}'(\pm1)=\mathrm{i}\alpha R\Phi_1'(\pm1)\,\Phi_1'(\pm1)-4\mathrm{i}\alpha R\Phi_{22}'(\pm1)-4\mathrm{i}\alpha\Phi_{33}(\pm1)\mp\Phi_3''(\pm1)+2RM^2\Phi_{22}'(\pm1)\mp\mathrm{i}\alpha\Phi_2'(\pm1)\mp\mathrm{i}\alpha R\Phi_1''(\pm1)-\mathrm{i}\alpha R\Phi_1(\pm1)\,\Phi_1''(\pm1)\pm RM^2\Phi_1''(\pm1),\quad(54)$$

where primes signify differentiation with respect to y.

The solutions of (37)–(42) are

$$\Phi_1(y)=A_1\sinh\alpha_1 y+B_1\sinh\beta_1 y,\quad(55)$$

$$\Phi_2(y)=A_2\cosh\alpha_1 y+B_2\cosh\beta_1 y,\quad(56)$$

$$\Phi_3(y)=A_3\sinh\alpha_1 y+B_3\sinh\beta_1 y,\quad(57)$$

$$\Phi_4(y)=-A_2\cosh\alpha_1 y-B_2\cosh\beta_1 y,\quad(58)$$

$$A_1=-\frac{(1+\lambda_1)\,R\delta}{(1-\mathrm{i}\alpha\lambda_2)\left(\beta_1^2-\alpha_1^2\right)\alpha_1\cosh\alpha_1},$$

$$B_1=\frac{(1+\lambda_1)\,R\delta}{(1-\mathrm{i}\alpha\lambda_2)\left(\beta_1^2-\alpha_1^2\right)\beta_1\cosh\beta_1},$$

$$A_2=\frac{2\mathrm{i}\alpha\alpha_1(1-\mathrm{i}\alpha\lambda_2)}{(1+\lambda_1)}A_1,$$

$$B_2=\frac{2\mathrm{i}\alpha\beta_1(1-\mathrm{i}\alpha\lambda_2)}{(1+\lambda_1)}B_1,$$

$$A_3=\frac{(1-\mathrm{i}\alpha\lambda_2)}{(1+\lambda_1)}\left(\alpha^2+\alpha_1^2\right)A_1,$$

$$B_3=\frac{(1-\mathrm{i}\alpha\lambda_2)}{(1+\lambda_1)}\left(\alpha^2+\beta_1^2\right)B_1,$$

$$\alpha_1^2=\frac{N+\sqrt{N^2-4\alpha\beta^2}}{2},$$

$$\beta_1^2=\frac{N-\sqrt{N^2-4\alpha\beta^2}}{2},$$

$$N=\alpha^2+\beta^2-\frac{\mathrm{i}\left(\alpha^2-\beta^2\right)M^2}{\alpha}.$$

The solution of (43)–(48) is

$$\Phi_{20}'(y)=F(y)+2C_1\left[\frac{\cosh\Gamma y-\cosh\Gamma}{\Gamma^2\cosh\Gamma}\right]+(D-F(1))\frac{\cosh\Gamma y}{\cosh\Gamma}.\quad(59)$$

The peristaltic mean flow is

$$\bar{u}(y)=\frac{\varepsilon^2}{2}\Phi_{20}'(y)=\frac{\varepsilon^2}{2}\left[F(y)+2C_1\left(\frac{\cosh\Gamma y-\cosh\Gamma}{\Gamma^2\cosh\Gamma}\right)+(D-F(1))\frac{\cosh\Gamma y}{\cosh\Gamma}\right],\quad(60)$$

$$D=-\frac{1}{2}\left[A_1\alpha_1^2\sinh\alpha_1+A_1^*\alpha_1^{*2}\sinh\alpha_1^*+B_1\beta_1^2\sinh\beta_1+B_1^*\beta_1^{*2}\sinh\beta_1^*\right],$$

$$C_1=-\frac{RM^2(1+\lambda_1)}{2}D+A_4\sinh\alpha_1+A_4^*\sinh\alpha_1^*+B_4\sinh\beta_1+B_4^*\sinh\beta_1^*,$$

$$A_4=-\frac{(1+\lambda_1)}{4}\left[\mathrm{i}\alpha\alpha_1^2RA_1+\mathrm{i}\alpha\alpha_1A_2+A_3\alpha_1^2\right],$$

$$B_4=-\frac{(1+\lambda_1)}{4}\left[\mathrm{i}\alpha\beta_1^2RB_1+\mathrm{i}\alpha\beta_1B_2+B_3\beta_1^2\right],$$

$$F(y)=s_1\cosh2\alpha y+s_2\cosh(\alpha+\beta)y+s_3\cosh(\alpha-\beta)y+s_4\cosh(\alpha+\beta^*)y+s_5\cosh(\alpha-\beta^*)y+s_6\cosh(\beta+\beta^*)y+s_7\cosh(\beta-\beta^*)y,$$

$$s_1 = \frac{\mathrm{i}\alpha A_1 A_1^* \left(\alpha_1^2 - \alpha_1^{*2}\right)}{4\left[\left(\alpha_1 + \alpha_1^*\right)^2 - \Gamma^2\right]} \cdot \left[R\left(1+\lambda_1\right) - \lambda_2 \left(\alpha_1 + \alpha_1^*\right)^2\right],$$

$$s_2 = \frac{\mathrm{i}\alpha A_1 A_1^* \left(\alpha_1^2 - \alpha_1^{*2}\right)}{4\left[\left(\alpha_1 - \alpha_1^*\right)^2 - \Gamma^2\right]} \cdot \left[-R\left(1+\lambda_1\right) + \lambda_2 \left(\alpha_1 - \alpha_1^*\right)^2\right],$$

$$s_3 = \frac{\mathrm{i}\alpha A_1 B_1^* \left(\alpha_1^2 - \beta_1^{*2}\right)}{4\left[\left(\alpha_1 + \beta_1^*\right)^2 - \Gamma^2\right]} \cdot \left[R\left(1+\lambda_1\right) - \lambda_2 \left(\alpha_1 + \beta_1^*\right)^2\right],$$

$$s_4 = \frac{\mathrm{i}\alpha A_1 B_1^* \left(\alpha_1^2 - \beta_1^{*2}\right)}{4\left[\left(\alpha_1 - \beta_1^*\right)^2 - \Gamma^2\right]} \cdot \left[-R\left(1+\lambda_1\right) + \lambda_2 \left(\alpha_1 - \beta_1^*\right)^2\right],$$

$$s_5 = \frac{\mathrm{i}\alpha B_1 A_1^* \left(\beta_1^2 - \alpha_1^{*2}\right)}{4\left[\left(\alpha_1^* + \beta_1\right)^2 - \Gamma^2\right]} \cdot \left[R\left(1+\lambda_1\right) - \lambda_2 \left(\alpha_1^* + \beta_1\right)^2\right],$$

$$s_6 = \frac{\mathrm{i}\alpha B_1 A_1^* \left(\alpha_1^{*2} - \beta_1^2\right)}{4\left[\left(\alpha_1^* - \beta_1\right)^2 - \Gamma^2\right]} \cdot \left[R\left(1+\lambda_1\right) - \lambda_2 \left(\beta_1 - \alpha_1^*\right)^2\right],$$

$$s_7 = \frac{\mathrm{i}\alpha B_1 B_1^* \left(\beta_1^2 - \beta_1^{*2}\right)}{4\left[\left(\beta_1 + \beta_1^*\right)^2 - \Gamma^2\right]} \cdot \left[R\left(1+\lambda_1\right) - \lambda_2 \left(\beta_1 + \beta_1^*\right)^2\right],$$

$$s_8 = \frac{\mathrm{i}\alpha B_1 B_1^* \left(\beta_1^2 - \beta_1^{*2}\right)}{4\left[\left(\beta_1 - \beta_1^*\right)^2 - \Gamma^2\right]} \cdot \left[-R\left(1+\lambda_1\right) + \lambda_2 \left(\beta_1 - \beta_1^*\right)^2\right].$$

At critical reflux condition, $\bar{u}$ is zero at $y = 0$ [29] and so invoking (60) we have

$$T_{\text{critical reflux}} = \frac{H_1 + \sqrt{H_1^2 - 4H_2}}{2}, \tag{61}$$

$$H = \frac{(h_1 + h_2)}{\alpha_1 \alpha_1^* \cosh\alpha_1 \cosh\alpha_1^*} + \frac{(h_3 + h_4)}{\alpha_1 \beta_1^* \cosh\alpha_1 \cosh\beta_1^*} + \frac{(h_5 + h_6)}{\alpha_1^* \beta_1 \cosh\alpha_1^* \cosh\beta_1} + \frac{(h_7 + h_8)}{\beta_1 \beta_1^* \cosh\beta_1 \cosh\beta_1^*},$$

$$H_1 = \frac{1}{\alpha^2}\left[2L_3 + g_3 + g_3^*\right],$$

$$H_2 = \frac{1}{\alpha^4}\left[L_4 L_4^* + L_4 g_3 + L_4^* g_3^*\right],$$

$$g = \frac{\mathrm{i}\alpha R^2 \left(1+\lambda_1\right)^2}{4\left(1+\alpha^2\lambda_2^2\right)\left(\beta_1^2 - \alpha_1^2\right)\left(\beta_1^{*2} - \alpha_1^{*2}\right)},$$

$$g_1 = -\frac{R\left(1+\lambda_1\right)}{2\left(1-\mathrm{i}\alpha\lambda_2\right)\left(\beta_1^2 - \alpha_1^2\right)}\left[\left(\frac{1-\cosh\Gamma}{\Gamma^2\cosh\Gamma}\right) \cdot \left(-\mathrm{i}\alpha R\left(1+\lambda_1\right) + \left(1-\mathrm{i}\alpha\lambda_2\right)\left(\alpha^2 - \alpha_1^2\right)\right) - 1\right],$$

$$g_2 = -\frac{R\left(1+\lambda_1\right)}{2\left(1-\mathrm{i}\alpha\lambda_2\right)\left(\beta_1^2 - \alpha_1^2\right)}\left[\left(\frac{1-\cosh\Gamma}{\Gamma^2\cosh\Gamma}\right) \cdot \left(\mathrm{i}\alpha R\left(1+\lambda_1\right) + \left(1-\mathrm{i}\alpha\lambda_2\right)\left(\beta_1^2 - \alpha^2\right)\right) - 1\right],$$

$$g_3 = -\frac{\mathrm{i}R^2}{\alpha H}\left[g_1\alpha_1\tanh\alpha_1 + g_2\beta_1\tanh\beta_1\right],$$

$$L_3 = \alpha^2 R^2 m - \alpha^4 B - K, \quad L_4 = \mathrm{i}\alpha R d + L_3,$$

$$h_1 = \frac{g\left(\alpha_1^2 - \alpha_1^{*2}\right)}{\left[\left(\alpha_1 + \alpha_1^*\right)^2 - \Gamma^2\right]}\left[1 - \frac{\cosh\left(\alpha_1 + \alpha_1^*\right)}{\cosh\Gamma}\right] \cdot \left[R\left(1+\lambda_1\right) - \lambda_2\left(\alpha_1 + \alpha_1^*\right)^2\right],$$

$$h_2 = \frac{g\left(\alpha_1^2 - \alpha_1^{*2}\right)}{\left[\left(\alpha_1 - \alpha_1^*\right)^2 - \Gamma^2\right]}\left[1 - \frac{\cosh\left(\alpha_1 - \alpha_1^*\right)}{\cosh\Gamma}\right] \cdot \left[-R\left(1+\lambda_1\right) + \lambda_2\left(\alpha_1 - \alpha_1^*\right)^2\right],$$

$$h_3 = \frac{-g\left(\alpha_1^2 - \beta_1^{*2}\right)}{\left[\left(\alpha_1 + \beta_1^*\right)^2 - \Gamma^2\right]}\left[1 - \frac{\cosh\left(\alpha_1 + \beta_1^*\right)}{\cosh\Gamma}\right] \cdot \left[R\left(1+\lambda_1\right) - \lambda_2\left(\alpha_1 + \beta_1^*\right)^2\right],$$

$$h_4 = \frac{-g\left(\alpha_1^2 - \beta_1^{*2}\right)}{\left[\left(\alpha_1 - \beta_1^*\right)^2 - \Gamma^2\right]}\left[1 - \frac{\cosh\left(\alpha_1 - \beta_1^*\right)}{\cosh\Gamma}\right] \cdot \left[-R\left(1+\lambda_1\right) + \lambda_2\left(\alpha_1 - \beta_1^*\right)^2\right],$$

$$h_5 = \frac{-g\left(\beta_1^2-\alpha_1^{*2}\right)}{\left[\left(\alpha_1^*+\beta_1\right)^2-\Gamma^2\right]}\left[1-\frac{\cosh\left(\alpha_1^*+\beta_1\right)}{\cosh\Gamma}\right] \cdot\left[R\left(1+\lambda_1\right)-\lambda_2\left(\alpha_1^*+\beta_1\right)^2\right],$$

$$h_6 = \frac{-g\left(\alpha_1^{*2}-\beta_1^2\right)}{\left[\left(\alpha_1^*-\beta_1\right)^2-\Gamma^2\right]}\left[1-\frac{\cosh\left(\alpha_1^*-\beta_1\right)}{\cosh\Gamma}\right] \cdot\left[R\left(1+\lambda_1\right)-\lambda_2\left(\beta_1-\alpha_1^*\right)^2\right],$$

$$h_7 = \frac{g\left(\beta_1^2-\beta_1^{*2}\right)}{\left[\left(\beta_1+\beta_1^*\right)^2-\Gamma^2\right]}\left[1-\frac{\cosh\left(\beta_1+\beta_1^*\right)}{\cosh\Gamma}\right] \cdot\left[R\left(1+\lambda_1\right)-\lambda_2\left(\beta_1+\beta_1^*\right)^2\right],$$

$$h_8 = \frac{g\left(\beta_1^2-\beta_1^{*2}\right)}{\left[\left(\beta_1-\beta_1^*\right)^2-\Gamma^2\right]}\left[1-\frac{\cosh\left(\beta_1-\beta_1^*\right)}{\cosh\Gamma}\right] \cdot\left[-R\left(1+\lambda_1\right)+\lambda_2\left(\beta_1-\beta_1^*\right)^2\right].$$

4. Results and Discussion

This section describes the effects of various emerging flow parameters. For this purpose, the mean-velocity at the boundaries of the channel, the time-averaged mean axial velocity distribution and reversal flow are calculated and the results are discussed graphically when $K_0 = 0$. The constant D which

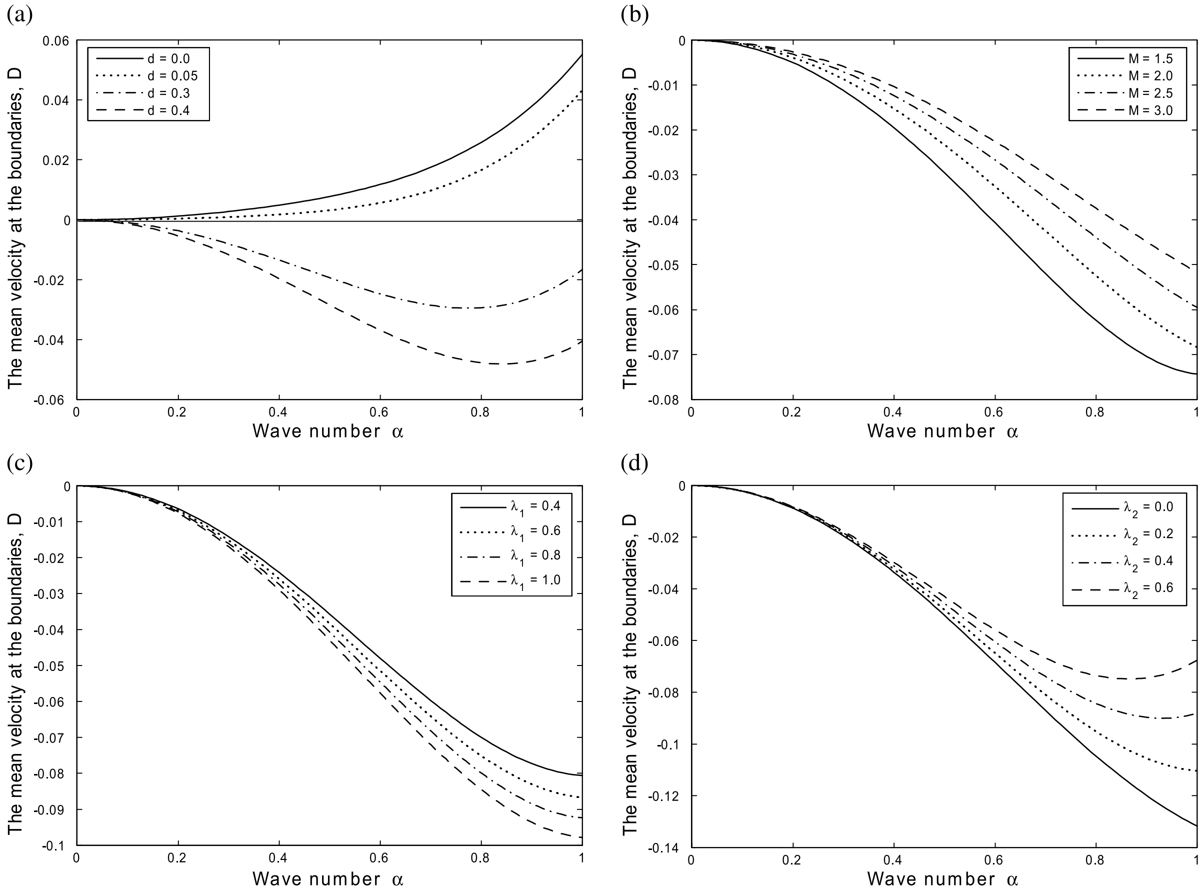

Fig. 2. Variation of D with wave number α for four different values of (a): the wall damping d, (b): Hartman number M, (c): the ratio of the relaxation time to the retardation time λ_1, and (d): the retardation time λ_2. The other parameters are chosen as: (a) $m = 0.01$, $B = 2$, $T = 1$, $K = 1$, $M = 2$, $R = 10$, $\lambda_1 = 0.8$, and $\lambda_2 = 0.5$; (b) $m = 0.01$, $B = 2$, $T = 1$, $K = 1$, $R = 10$, $d = 0.5$, $\lambda_1 = 0.8$, and $\lambda_2 = 0.4$; (c) $m = 0.01$, $B = 2$, $T = 1$, $K = 1$, $R = 10$, $M = 1$, $d = 0.5$, and $\lambda_2 = 0.2$; (d) $m = 0.01$, $B = 2$, $T = 1$, $K = 1$, $R = 10$, $M = 1$, $d = 0.5$, and $\lambda_1 = 1.5$.

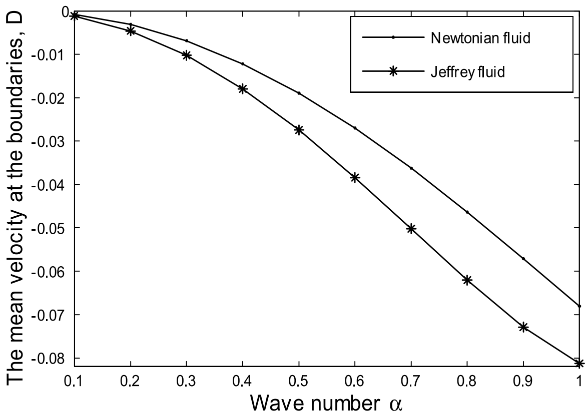

Fig. 3. Variation of D with wave number α for different fluids. The values of the other parameters are $m = 0.01$, $B = 2$, $T = 1$, $K = 1$, $M = 2$, $R = 10$, and $d = 0.5$.

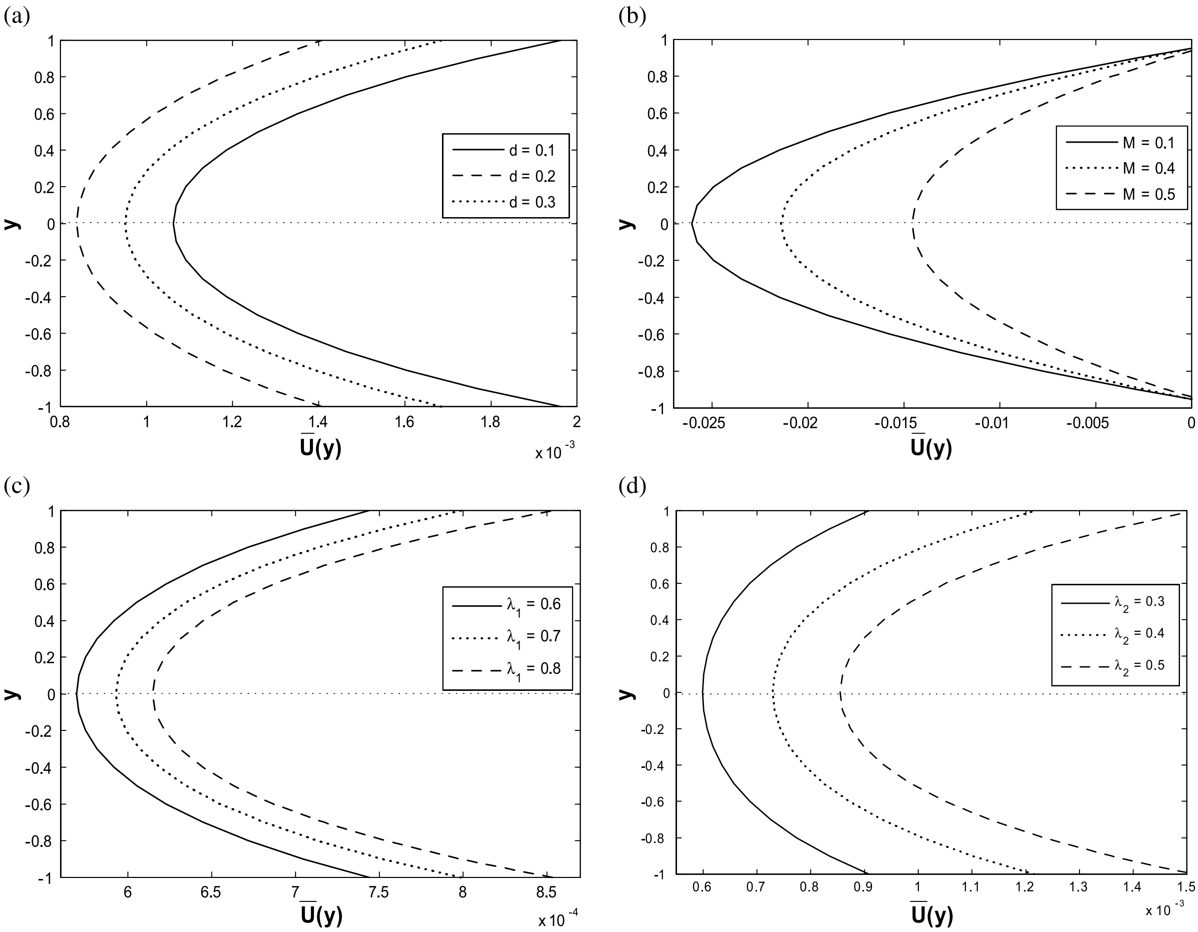

Fig. 4. Variation of mean-velocity distribution and reversal flow for three different values of (a): the wall damping d, (b): Hartman number M, (c): the ratio of the relaxation time to the retardation time λ_1, and (d): the retardation time λ_2. The values of other parameters are: (a) $m = 0.01$, $B = 2$, $T = 1$, $K = 1$, $R = 1$, $\alpha = 0.5$, $\varepsilon = 0.15$, $\lambda_1 = 0.8$, and $\lambda_2 = 0.4$; (b) $m = 0.01$, $B = 2$, $K = 1$, $T = 1$, $R = 1$, $d = 0.5$, $\alpha = 0.5$, $\varepsilon = 0.15$, $\lambda_1 = 0.8$, and $\lambda_2 = 0.4$; (c) $m = 0.01$, $B = 2$, $T = 1$, $K = 1$, $R = 1$, $M = 1$, $d = 0.5$, $\alpha = 0.5$, $\varepsilon = 0.15$, and $\lambda_2 = 0.4$, and (d) $m = 0.01$, $B = 2$, $T = 1$, $K = 1$, $R = 1$, $M = 1$, $d = 0.5$, $\alpha = 0.5$, $\varepsilon = 0.15$, and $\lambda_1 = 0.4$.

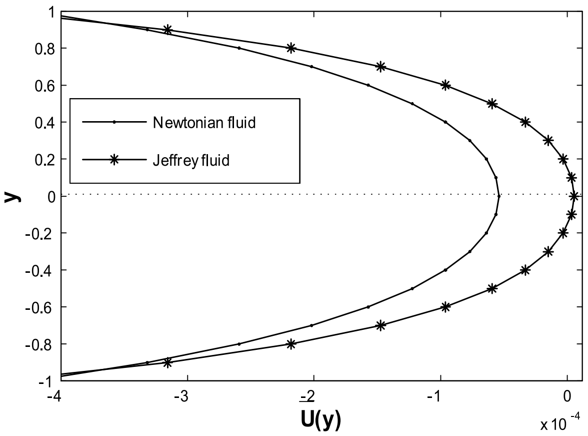

Fig. 5. Variation of mean-velocity distribution and reversal flow for different fluids. The values of other parameters are $m = 0.01$, $B = 2$, $T = 1$, $K = 1$, $M = 1$, $R = 10$, $\alpha = 0.5$, $\varepsilon = 0.15$, and $d = 0.5$.

initially arises from the no-slip condition of the axial-velocity on the wall is due to the value of $\Phi'_{20}(y)$ at the boundary. The mean-velocity at the boundaries of the channel is $\bar{u}(\pm 1) = \frac{\varepsilon^2}{2}\Phi'_{20}(\pm 1) = \frac{\varepsilon^2}{2}D$ [29]. Figure 2a depicts wall damping (d) effects on the boundaries D. We conclude that D decreases upon increasing d. The effects of Hartman number M on the mean velocity at the boundaries are shown in Figure 2b. It is found from this figure that the mean velocity at the boundaries increases with increasing Hartman number M. Figure 2c elucidates the effect of the ratio of the relaxation time to the retardation time λ_1 on the mean velocity at the boundaries of the channel. It is noticed that D decreases with an increase in λ_1. However, an opposite behaviour is observed in Figure 2d which shows the effect of the retardation time λ_2 on D. The effects

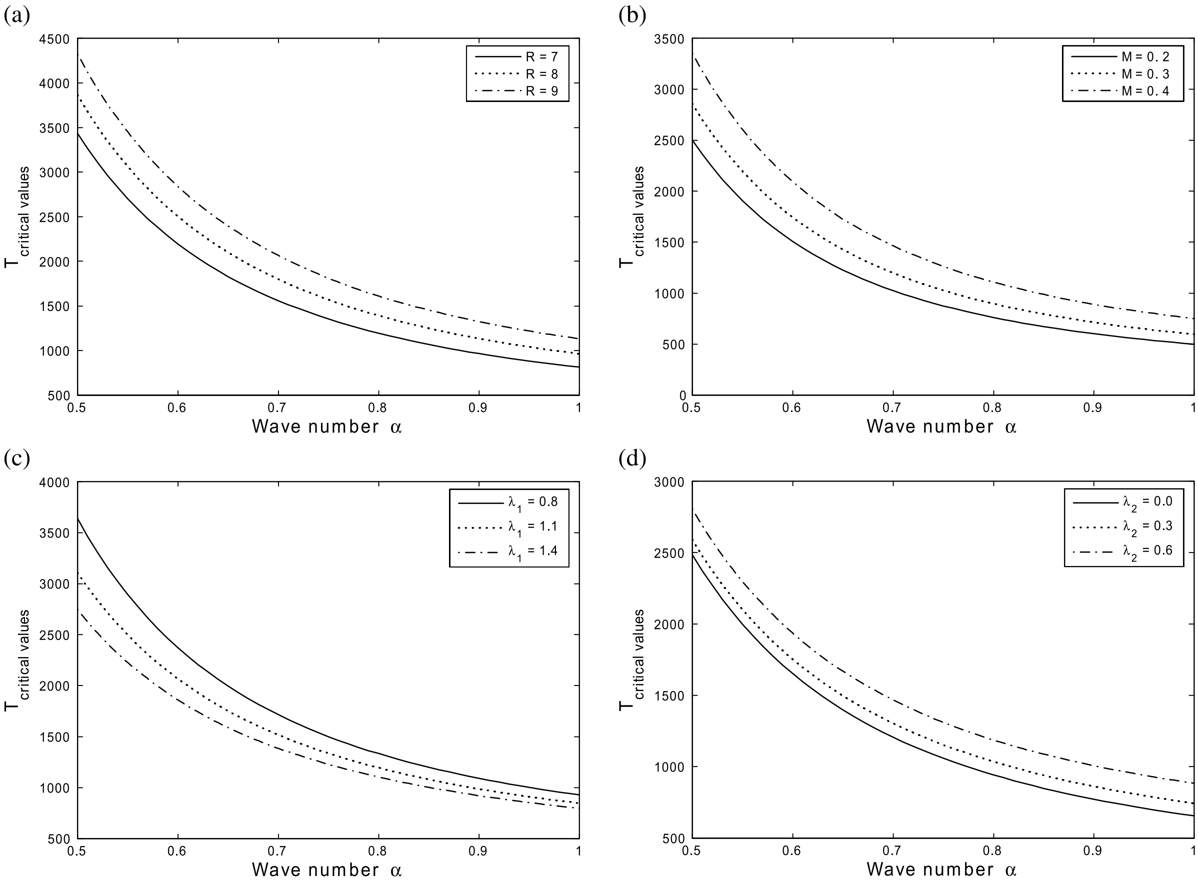

Fig. 6. Variation of critical values of wall tension T with wave number α for three different values of (a): the Reynolds number R, (b): Hartman number M, (c): the ratio of the relaxation time to the retardation time λ_1, and (d): the retardation time λ_2. The values of other parameters are: (a) $m = 0.01$, $B = 2$, $K = 1$, $d = 0.5$, $M = 0.5$, $\lambda_1 = 0.7$, and $\lambda_2 = 0.4$; (b) $m = 0.01$, $B = 2$, $K = 1$, $d = 0.5$, $R = 8$, $\lambda_2 = 0.4$, and $\lambda_1 = 0.7$; (c) $m = 0.01$, $B = 2$, $K = 1$, $R = 8$, $M = 0.5$, $d = 0.5$, and $\lambda_2 = 0.4$, and (d) $m = 0.01$, $B = 2$, $K = 1$, $R = 8$, $M = 0.5$, $d = 0.5$, and $\lambda_1 = 1.5$.

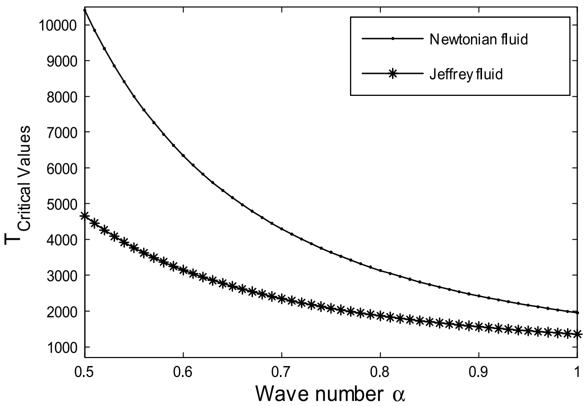

Fig. 7. Variation of critical values of wall tension T with wave number α for different fluids. The values of other parameters are $m = 0.01$, $B = 2$, $K = 1$, $M = 0.5$, $R = 10$, $\alpha = 0.5$, and $d = 0.5$.

of different fluids on the mean velocity at the boundaries of the channel are depicted in Figure 3. This figure shows that the mean velocity at the boundaries is greater in the case of the Newtonian fluids when compared to a Jeffrey fluid. It is noted that damping may cause the mean flow reversal at the walls which is not possible in the elastic case. Furthermore, for small α the damping occurs which indicate some disturbances during the motion. The variations of d and M on the mean velocity distribution and reversal flow are plotted in Figures 4a and b. It is revealed that the possibility of flow reversal increases by increasing M (Fig. 4b) while in case of d the situation is reversed (Fig. 4a). The variations of λ_1 and λ_2 on the mean velocity distribution and reversal flow are shown in the Figures 4c and d. It is noticed that the flow reversal increases by increasing λ_1 and λ_2 (Figs. 4c and d). The behaviour of different fluids on the mean velocity distribution and reversal flow is depicted in Figure 5. This figure indicates that magnitude of the velocity is large for the linear Jeffrey fluid when compared with the Newtonian fluid. Following Fung and Yih [31], the mean-velocity perturbation function $G(y)$ is defined by

$$G(y) = -\frac{200}{\alpha^2 R^2}\left[F(y) - F(1)\right]. \qquad (62)$$

From Figures 6a and b it is observed that the critical value of T increases when Reynolds number R and the Hartman number M are increased. It is seen from Figure 6c that the effect of the ratio of the relaxation time to the retardation time λ_1 on critical value of T is quite opposite to that of λ_2 (Fig. 6d). We further note that the critical value of T is very high for small values of the wave number α when compared with its large values. Figure 7 reveals the fact that the critical values of T are greater in comparison of the Newtonian fluid with Jeffrey fluid.

5. Closing Remarks

This study describes the peristaltic flow of an incompressible Jeffrey fluid. The two-dimensional equations are modeled and the analytical results are presented for the free pumping case. The main results of this analysis are as follows:

- There is a decrease in the constant D when λ_1 is increased.
- The role of λ_2 on D is quite opposite to that of λ_1.
- The flow reversal increases by increasing λ_1 and λ_2 while the flow reversal decreases when d increases.
- The magnitude of velocity in Jeffrey fluid is greater than that of Newtonian fluid.
- The corresponding results of viscous fluid [29] can be deduced when λ_1 and λ_2 are equal to zero.

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